T. Ekholm – 29/5/08 – A surgery long exact sequence in contact homology

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- **Linearized contact homology**

Recall: Y contact manifold $\rightsquigarrow (Q(Y), d)$ DG-algebra

Def: an augmentation is an $\begin{cases} \text{algebra homomorphism} \\ \text{chain map} \end{cases}$ $\varepsilon : Q(Y) \to \mathbb{Q}$

(in particular: $\begin{cases} \varepsilon(1) = 1 \\ \varepsilon \circ d = 0 \end{cases}$; ε is determined by its value on generators)

$$Q(Y) = \underset{\text{constants}}{Q_0(Y)} \oplus \underset{\text{linear}}{Q_1(Y)} \oplus Q_{\geq 2}(Y)$$

Ex. $\varepsilon(1) = 1$, $\varepsilon(\text{generators}) = 0$ is an augmentation iff $d(\text{generators})$ have no constant terms, ie. $d(Q(Y)) \subseteq Q_{\geq 1}(Y)$

Then $d_1 : Q_1(Y) = Q_{\geq 1}(Y) / Q_{\geq 2}(Y) \circlearrowleft$ linearized differential.

In general, if $\varepsilon : Q(Y) \to \mathbb{Q}$ is an augmentation, let

$$E_\varepsilon(\gamma) := \gamma + \varepsilon(\gamma) \quad \text{and} \quad d^\varepsilon = E_\varepsilon \circ d \circ E_\varepsilon^{-1}.$$

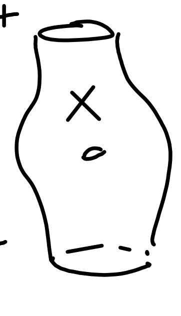
Then d^ε has the property that it maps Q to $Q_{\geq 1}$, and induces a linearized differential $d_1^\varepsilon : Q_1(Y) \to Q_1(Y)$

Define $LCH(Y; \varepsilon) := \ker d_1^\varepsilon / \operatorname{Im} d_1^\varepsilon$ linearized contact homology

Note: if $\varepsilon_0, \varepsilon_1$ are homotopic, ie. $\varepsilon_1 - \varepsilon_0 = K \circ d$, then LCH's are isomorphic.

- **SFT:**

Y^+



Y^-

$\rightsquigarrow$

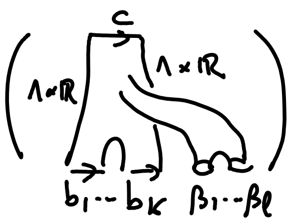
$$\begin{array}{c} Q(Y_+) \\ \big\downarrow \Phi_X \ \text{chain map} \\ Q(Y_-) \underset{\text{given}}{\overset{\varepsilon}{\Rightarrow}} \mathbb{Q} \end{array} \Rightarrow \text{induces} \quad \begin{array}{c} Q_1(Y^+; \varepsilon \circ \Phi_X) \\ \big\downarrow \Phi_X^1 \ \text{chain map} \\ Q_1(Y_-; \varepsilon) \end{array}$$

In particular, a symplectic filling of Y induces an augmentation.

- <u>Legendrian contact homology</u> in presence of an augmentation:

  $A(Y,\Lambda;\varepsilon) \circlearrowleft \partial$ algebra over $\mathbb{Q}$ with differential

$$\partial c = \sum \# \mathcal{M}\Big(\;\; \Big)\, \varepsilon(\beta_1 \dots \beta_\ell)\, b_1 \dots b_k$$



Ie: $A(Y,\Lambda;\varepsilon) = A(Y,\Lambda) \underset{\mathbb{Q}(Y)}{\otimes} \mathbb{Q}$

$A(Y,\Lambda;\varepsilon) = A_0 \oplus A_{\geq 1} \rightsquigarrow$ let $A^+ = A/A_0$. $\circlearrowleft$ induced ∂^+

- <u>cyclic version</u>: $A^c = A^+/\sim$ (NOT AN ALGEBRA!)

  $c_1 \cdots c_r \sim (-1)^{\varepsilon(\sigma)} c_{\sigma(1)} \cdots c_{\sigma(r)}$

  for σ cyclic permutation

  Then ∂^+ on A^+ induces a differential $\partial^c : A^c \circlearrowleft$

  (<u>Note:</u> in A^c, $\deg(a)$ odd $\Rightarrow a^2 = 0$)

- <u>linearized version</u>:

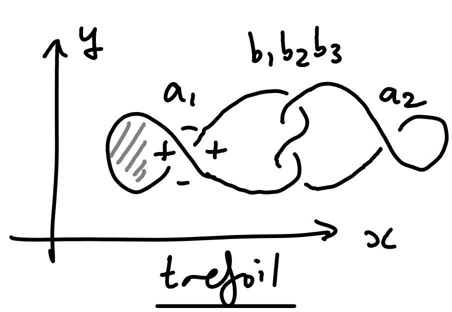
  $\Lambda \subset Y = \mathbb{R}^3$ Legendrian bounds L Lagrangian

  $\Rightarrow$ induces an augmentation $\varepsilon : A(\Lambda) \to \mathbb{Z}_2$

  Given ε augmentation (geometric or not) $\rightsquigarrow$ linearized $A_1(\Lambda,\varepsilon)$

<u>Ex:</u>



$\partial a_1 = 1 + b_3 + b_1 + b_3 b_2 b_1$ $\quad |a_i| = 1$

$\partial a_2 = 1 + b_3 + b_1 + b_1 b_2 b_3$ $\quad |b_i| = 0$

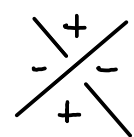
$\partial b_1 = \partial b_2 = \partial b_3 = 0$

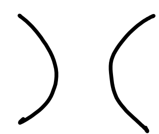
$\rightsquigarrow \exists$ 5 augmentations: $\varepsilon(a_i) = 0$ for degree reasons

	b_1	b_2	b_3
ε_1	1	0	0
ε_2	0	0	1
ε_3	1	1	1
ε_4	1	1	0
ε_5	0	1	1

- Can try to realize them by Lagrangian fillings / cobordisms.

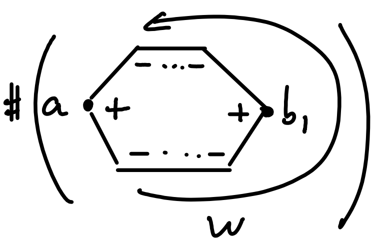
<u>Idea:</u> look at chain maps from cobordisms given by smoothing crossings:

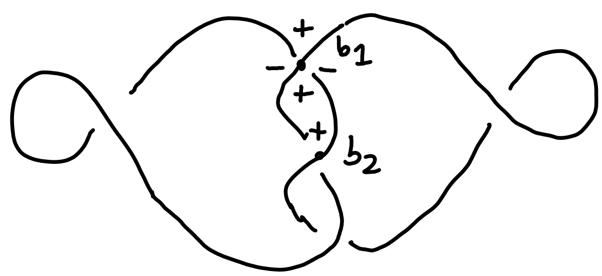
e.g. smooth b_1 

$\rightsquigarrow$ 

[also do:  and  $\to \emptyset$]

gives chain map $b_1 \longmapsto 1$

$a \longmapsto a + \sum_{\text{holom. discs}} \#$ 

ie. here 

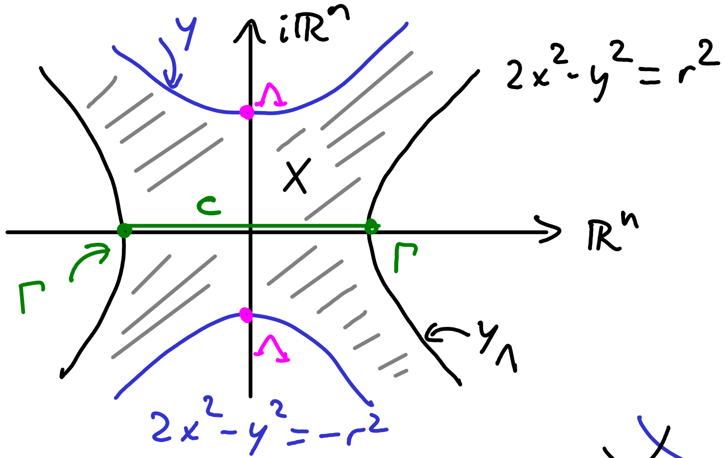
Smoothing b_1 gives $b_1 \mapsto 1$ & identity on others
$b_2 \mapsto b_2 + 1$

$\rightsquigarrow$ Compute inductively augmentations from possible Lagr. cobordisms that eventually give a filling of Λ.

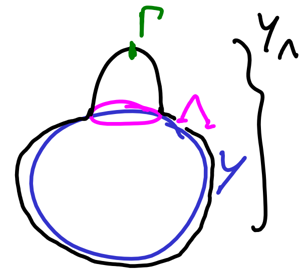
(here: 5 non-homotopic augmentations $\leftrightsquigarrow$ 5 non-isotopic Lagr. fillings)

Legendrian surgery: $\Lambda \subset Y$ Legendrian S^{n-1}

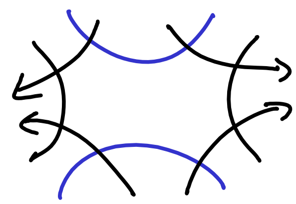
$\Rightarrow$ attach a handle:



$\rightsquigarrow$ attach:



Liouville v.f.
$v = 2x\,\partial_x - y\,\partial_y$



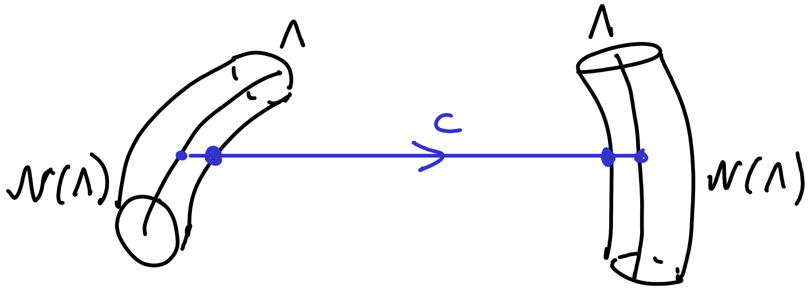
(The handle looks like ST^*D^n, with core Γ = fiber over 0)

<u>NB:</u> $\exists$ sympl. cobordism X between Y and Y_Λ $\rightsquigarrow$ maps on contact H_*.

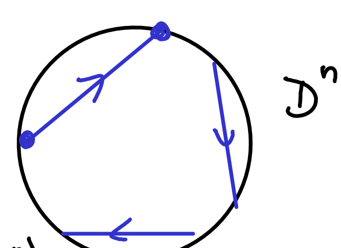
<u>Goal:</u> compute $\begin{cases} \text{the linearized CH of } Y_\Lambda \\ \text{the linearized or full CH of } \Gamma = \text{core of handle:} \\ \qquad \text{Legendrian} \subset Y_\Lambda. \end{cases}$

$\rightarrow$ need to understand <u>Reeb dynamics in Y_Λ</u>

- A Reeb orbit in Y_Λ consists of Reeb chords $\Lambda \rightsquigarrow \Lambda$



together with pieces in the new handle



$\star$ can always match these...
(Reeb dynamics on ST^*D^n = geodesic flow
$\Rightarrow \exists!$ chord b/w 2 given pts of $S^{n-1} = \partial D^n$)

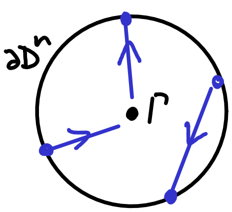
→ **Claim:** Reeb orbits in Y_Λ correspond to

1) Reeb orbits in Y
2) cyclic words of Reeb chords of Λ.

Similarly: Reeb chords of $\Gamma \longleftrightarrow$ words of Reeb chords of Λ

because: in D^n, looks like

$\Lambda \leftrightarrow \partial D^n$



$+$ chords in (Y, Λ)

- given augmentation ε on $Q(Y)$, $\Lambda \subset Y$ Legendrian:

Thm 1: $\exists$ long exact sequence

$$CCH_{s+(n-3)}(Y, \Lambda; \varepsilon) \xrightarrow{\iota} LCH_s(Y_\Lambda, \varepsilon_\Lambda) \xrightarrow{\Phi_X} LCH_s(Y, \varepsilon) \to \cdots$$

cyclic Leg. contact homology

$\varepsilon_\Lambda = \varepsilon \circ \phi_X$ induced by cobordism $Y \to Y_\Lambda$

Thm 2:

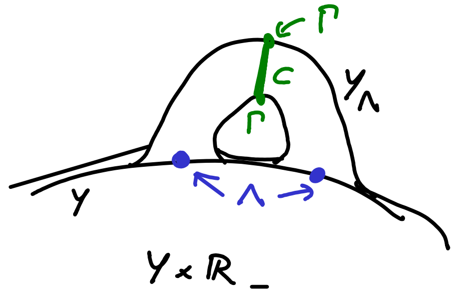
$$LCH_s(Y_\Lambda, \Gamma; \varepsilon_\Lambda, \varepsilon_C) \xrightarrow[\rho]{\cong} RCH_{s+(n-2)}(Y, \Lambda; \varepsilon)$$

linearized Leg. contact homology using $\begin{cases} \varepsilon_\Lambda \text{ on } Q(Y_\Lambda) \\ \varepsilon_C \text{ induced by co-core } C \text{ of handle } \partial C = \Gamma \end{cases}$

$= \ker(d : A^+ \to A^+)/\mathrm{Im}$
reduced Leg. contact homology

Thm 3: $\exists$ DGA $B(Y, \Lambda; \varepsilon)$ generated by words of Reeb chords with a differential Δ defined by a curve count in $(Y \times \mathbb{R}, \Lambda)$ and a <u>DGA isomorphism</u> $A(Y_\Lambda, \Gamma; \varepsilon_\Lambda, d^{\varepsilon_C}) \xrightarrow{\varrho} B(Y, \Lambda; \varepsilon)$

ρ is defined using curves in



positive end = one chord in (Y_Λ, Γ)
neg. end = collection of chords in (Y, Λ).

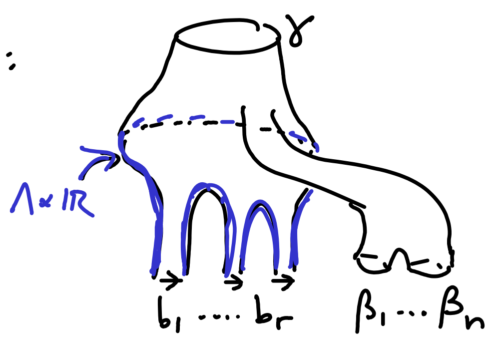
<u>To prove Thm 1</u>, take $C(Y, \Lambda; \varepsilon)$ chain complex generated by Reeb orbits and by cyclic words in Reeb chords

$\delta : C \to C$ defined by

- $\delta w = \partial^c w$ for a cyclic word w
  (∂^c = diff! on cyclic relative $A^c(Y, \Lambda, \varepsilon)$
- $\delta \gamma = \delta^{(0)} \gamma + \delta^{(1)} \gamma$ for a Reeb orbit γ

$$\left\{ \begin{array}{ll} \delta^{(0)} \gamma = d\gamma, & d : Q_1(Y, \varepsilon) \circlearrowleft \text{ linearized CH differential} \\[1em] \delta^{(1)} \gamma = \displaystyle\sum_{\dim \mathcal{M}(\gamma; \overline{\beta}, \overline{b}) = 1} K(\overline{\beta})^{-1} \cdot \left| \mathcal{M}/\mathbb{R} \right| \cdot \varepsilon(\overline{\beta}) \cdot \overline{b} & \end{array} \right.$$

in $Y \times \mathbb{R}$:



$K(\overline{\beta}) = \prod K_{\beta_i}$
multiplicities of orbits
$\varepsilon(\overline{\beta}) = \prod \varepsilon(\beta_i)$

Need to check δ is a differential ...

[NDDA: $b_1 \ldots b_r$ are only <u>cyclically</u> ordered, hence fits in with $A^c(Y, \Lambda)$]